# Mathematical analysis

## Program

**Clemente Zanco, 2020/10/26 - 2020/11/11.**

(i) Introduction to the course and plan of lectures. Motivation for studying sequences and series of functions. Basic definitions. Pointwise convergence and properties that it does not keep, counterexamples. Uniform convergence and uniform Cauchy condition. Characterization of the Riemann integrable bounded functions by mean of the size of the set of their points of non-continuity. Uniform convergence preserves boundedness, continuity, integrability and integral.

(ii) Differentiation of sequences. Series of functions and related criteria; special case of Leibniz series.

(iii) Concept and definition of metric space. Discs in metric spaces; signficant examples of metric spaces. Subspaces of metric spaces: induced metric. Topology induced by a metric, basic topological concepts and definitions in metric spaces; relevant examples. Completeness.

(iv) Closure and completeness, density. Linear structure, metrics and norms, normed spaces. Examples of non-complete normed spaces. Banach spaces. The completion theorem. The space of the continuous functions on $[0, 1]$ and the space of the polinomials; the space of the continuously differentiable functions with the maximum norm and with the natural norm.

(v) Convergence in $\mathbb{R}^d$ in any norm is convergence by coordinates. Space of the continuous functions of compact support in $\mathbb{R}$: maximum norm and integral norm, these two norms do not induce nested topologies and no of them is complete. The integral as a linear functional. Completion in the maximum norm, motivation for introducing Lebesgue measure in $\mathbb{R}$.

A suitable bibliography will be suggested depending on the audience. In any case, the following is valuable.

## References

[1] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill.
[2] N. Fusco, P. Marcellini, C. Sbordone, *Analisi Matematica due*, Liguori Editore (only for students who can read italian).

**Carlo Alberto De Bernardi, 2020/11/12 - 2020/12/01.**

(i) $\sigma$-algebras and Borel $\sigma$-algebras with respect to a given topology. Measurable functions with values in a metric space; comments on the definition. Criteria for a function with real extended values to be measurable. Operations that do not lead measurability; composition with continuous functions. Simple functions and approximation by them.

(ii) Positive measures: definition, elementary properties and elementary significant examples. Properties that hold almost everywhere with respect to a given measure. Completion of a measure. The Lebesgue measure on $\mathbb{R}$. Integration of positive functions with respect to a generic positive measure: monotone convergence theorem, Fatou's lemma, basic properties of the integral.

(iii) Integration of real functions; the integral as a linear functional on the linear space of integrable functions. Dominated convergence theorem. The fundamental theorem on integration by series. The space $L_1(\mu)$ as a Banach space and the integral as a continuous linear functional on it.

(iv) Relationships between Lebesgue and Riemann proper and improper integral. The special case of the space $\mathbb{R}^d$ as $L_1(\mu_c)$ space constructed starting from a set of $d$ points under the counting measure $\mu_c$.

(v) Exercises on $\sigma$-algebras. Exercises on the Lebesgue integration in $\mathbb{R}$.

References

[1] H.L. Royden, *Real Analysis*, Macmillan. (Chapters 3,4)
[2] W. Rudin, *Real and Complex Analysis*, McGraw-Hill. (Chapter 1)

**Enrico Miglierina, 2020/12/04 - 2020/12/21.**

(i) Product measure. Integration on product spaces.

(ii) Tonelli and Fubini theorems. Some counterexamples.

(iii) $\mathcal{L}^p(X, \Sigma, \mu)$ spaces and $L^p(X, \Sigma, \mu)$. Hölder inequality and Minkowski inequality. $L^p(X, \Sigma, \mu)$ are Banach spaces for $1 \leq p \leq \infty$. An example: $\ell^p$.

(iv) Inner product spaces: definition and basic properties. Cauchy-Schwarz inequality. Norms induced by an inner product (parallelogram and polarization identities). Hilbert spaces: definition and some examples ($L^p(X, \Sigma, \mu)$, $\ell^2$, $\ell^2(\Gamma)$. The notion of orthogonality. Best approximation Theorem and Projection Theorem.

(v) Orthonormal systems. Complete orthonormal systems. Separable metric space Separable Hilbert space has a countable (at most) complete orthonormal system (Gram -Schmidt orthonormalization). Orthonormal system in $\ell^2$ and $\ell^2(\Gamma)$. Fourier coefficients. Bessel inequality and Parseval Identity. Riesz- Fischer Theorem. Isometry with $\ell^2(\Gamma)$.

## References


[1] W. Rudin, *Real and Complex Analysis*, McGraw-Hill. (Chapters 3,4,7)

[2] Lecture notes.