Probability 1 & 2

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PROGRAM.

- **Axioms of probability.** The point of view of Kolmogorov and de Finetti: σ-additivity and finite additivity. Definition of a probability space, the σ-additivity and its equivalent formulation as a continuity property of the probability measure. The elementary properties of probability: finite additivity; the probability is monotone; the inclusion-exclusion formula. The consequences of σ-additivity: continuity from below, continuity from above and Boole's inequality. Example: the problem of the matching experiment.

- **Extensions of probability measures.** Definitions: π-systems and λ-systems, the $\pi-\lambda$ Theorem. Extension theorem (see [1, Theorem 3.1, pg.36]): proof of uniqueness. The Lebesgue measure on $(0, 1)$.

- **Denumerable probabilities.** Independence of events: the elementary notion, elementary definition of conditional probability. Example: the problem of Monty Hall. Independence of a family of events, independence of n classes of events. Almost sure and almost impossible events. Definition of limsup and liminf of a sequence of events. The Borel-Cantelli lemmas and the example of the monkey. The tail σ-field. The Kolmogorov 0-1 law.

- **Random variables and expectations.** Definition of a Polish space. Definition of a random variable and its meaning. The σ-algebra generated by a random variable. The probability distribution of a random variable. Independence of random variables: definition using the generated σ-algebra. The Lebesgue integral: definition, integrable functions, change of variable theorem (proof using standard machinery: sketch). Expectation of a random variable: the general definition and its properties. The case of discrete and absolutely continuous random variables. Properties of the expected value: linearity, monotonicity, etc. $\mathscr{L}^p$ and L^p spaces, the Hölder inequality. Monotone convergence theorem, Dominated convergence theorem, Fatou's Lemma. Application of the monotone convergence theorem to study the convergence of a series of positive random elements. Example: the Cantelli inequality.

- **Conditional expectations and probabilities.** The Radon-Nikodym theorem and the Lebesgue decomposition theorem. The conditional expectation given a σ-field: definitions and main properties. Some examples. Conditional probabilities: definition and main properties.

- **Convergences of random variables.** Almost sure convergence, in L^p and in probability. Convergence in probability of a vector is equivalent to the convergence of the components, characterization of the convergence in probability. The relations among the different modes of convergence: implications and counterexamples. The continuous mapping theorem for convergence in probability. Partial converses. Examples and exercises.

- **Weak convergence and convergence in distribution.** The definition of weak convergence of measures. Convergence in distribution of sequences of random variables. Equivalent characterizations of convergence in distribution. The relations between convergence in distribution and the other modes: convergence in probability implies convergence in distribution, but the converse is not true. Partial converse: the convergence in distribution to a degenerate random variable implies the convergence in probability. Continuous mapping theorem for the convergence in probability and in distribution: proofs. The general formulation of the Slutsky theorem and its applications. Examples and exercises.

- **Characteristic functions and weak convergence of probability measures: the main theorems.** Characteristic functions: definition and its properties. The Lévy inversion theorem [1, Theorem 26.2] and the Lévy continuity theorem [1, Theorem 26.3]. The moment generating function and the probability generating function.

- **Limit theorems in probability: weak convergence.** Central limit theorems: Lindeberg-Lévy theorem, Lindeberg and Lyapounov theorems. An application: the Delta method. Examples and exercises.

- **Laws of large numbers.** The weak law of large numbers. Strong laws of large numbers: statements. The version of the strong law due to Etemadi. Some comments on the proof of the SLLN by Kolmogorov. A SLLN for sequences of independent random variables.

TEACHING METHODS. The teaching methods consist of traditional lessons and class exercises. The lectures require an interactive participation of the Ph.D. students, who will be asked, e.g., to solve exercises assigned during the course.

References

[1] Billingsley, P. (1995). *Probability and measure.* John Wiley & Sons, New York.

[2] Jacod, J. and Protter, P. (2004). *Probability essentials.* Springer, Berlin.

[3] Resnick, S.I. (2014). *A probability path.* Birkhäuser/Springer, New York.

[4] Williams, D. (1999). *Probability with martingales.* Cambridge University Press, Cambridge.